# 8) Non-repelling vs Repelling cycles

In the previous chapter, we used the information given by local dynamics, and showed that in the basin of attraction of each (each) contracting or parabolic cycle lies a critical point. Being the number of critical points of a rational map in $\hat{\mathbb{C}}$ of degree $d \geq 2$ bounded (by $2d-2$), and different basins of attraction disjoint, we deduced that the number of contracting or parabolic cycles is bounded by $2d-2$.

In the neutral case, the control is more difficult to achieve: Siegel disks do not contain critical points, while we only know that Cremer points lie in the post-critical set. The next theorem achieves an estimate on the number of indifferent cycles.

Theorem: For a rational map $f: \hat{\mathbb{C}} \circlearrowleft$ of degree $d \geq 2$, the number of indifferent cycles of multiplier $\lambda \neq 1$ is at most $4d-4$

Proof. The idea (Fatou) is to perturb the given map $f$, in a way that at least half of its indifferent cycles become attracting, which gives the wanted estimate. Write $f(z) = \frac{P(z)}{Q(z)}$, $P, Q$ without common factors,

and $\max\{\deg P, \deg Q\} = d$.

Consider the one parameter family of maps:

$f_t(z) = \dfrac{P(z) - t z^d}{Q(z) - t}$ $t \in \hat{\mathbb{C}}$. For $t=0$ we get $f_0 = f$, for $t = \infty$ we get $f_\infty(z) = z^d$.

Assume $f(z)$ is not itself $z^d$ (in which case there are no cycles but for the fixed points at $0$ and $\infty$ (superattracting)).

But for finitely many values of $t$, $f_t$ is a rational map of degree $d$. In fact, the degree may drop only if there is $z$ so that $P(z) - tz^d = Q(z) - t = 0$, which happens if and only if $f(z) = z^d$, which has finitely many solutions

and for any such $z$, $b = Q(z)$, which gives finitely many $b$;
one if $\deg P > \deg Q$, and $b = a_d$, $a_d$ the leading coefficient of $P$.

In particular, for $|b| << 1$, $f_b$ has degree $d$.

Suppose now that $f_0$ has $k$ distinct indifferent cycles (of multiplier $\lambda_j \neq 1$). To prove that $k \leq 4d-4$, we will show that there exists $b$, $|b| << 1$, so that $f_b$ has at least $\frac{k}{2}$ attracting cycles.
This proves the theorem.

For any cycle, pick a point $z_j$ in the cycle ($j = 1 \ldots k$). Denote by $m_j$ the period of the cycle, so that $f^{m_j}(z_j) = z_j$ and $(f^{m_j})'(z_j) = \lambda_j$, $|\lambda_j| = 1$, $\lambda_j \neq 1$.

By the implicit function theorem (applied to $f^{m_j}(z) - z$ at $z = z_j$, $b = 0$), we can follow such cycles under small perturbation of $f_0$ to $f_b$ (because $\lambda_j \neq 1$).
I.e., for $|b| << 1$, we have $z_j(b)$ satisfying $f_b^{m_j}(z_j(b)) = z_j(b)$.
Moreover, its multiplier $\lambda_j(b) = (f_b^{m_j})'(z_j(b))$ varies holomorphically on $b$.

Claim: None of the functions $b \mapsto \lambda_j(b)$ is locally constant at $b = 0$.
(We prove this claim later).

Hence we can write $\lambda_j(b) = \lambda_j(1 + \ell_j b^{r_j} + o(b^{r_j}))$, $r_j \neq 0$

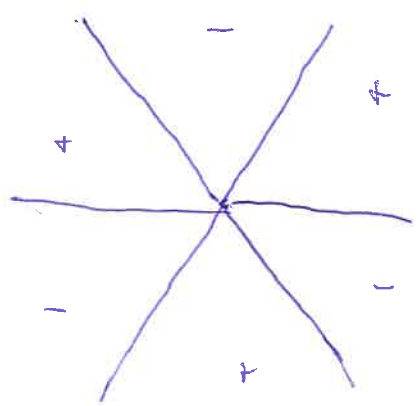
In particular, $|\lambda_j(b)| = 1 + \mathrm{Re}(\ell_j b^{r_j}) + o(b^{r_j})$

We can divide the $b$-plane in $r_j$ sectors where $\mathrm{Re}(\ell_j b^{r_j})$ is positive, and $r_j$ complementary sectors where $\mathrm{Re}(\ell_j b^{r_j})$ is negative

Set $\sigma_j(\theta) = \mathrm{sign}(\mathrm{Re}(\ell_j e^{2\pi i \theta r_j}))$, so that

$\sigma_j(\theta) = 1 \Rightarrow |\lambda_j(\rho e^{2\pi i \theta})| > 1$, $\rho << 1$

" $= -1$ " $< 1$ "




The maps $\sigma_j : \mathbb{R}/\mathbb{Z} \to \{-1, 0, 1\}$ is a step function with $2r_j$ discontinuities, and average $0$.

Set $\check{k} = \begin{cases} k & \text{if } k \text{ odd} \\ k-1 & \text{if } k \text{ even.} \end{cases}$ Then $\sum_{j=1}^{\check{k}} \sigma_j$ also has average zero, and

takes odd values almost everywhere. In particular we can choose $\theta$ so that

at least $\left\lceil \frac{\check{k}}{2} \right\rceil = \frac{\check{k}+1}{2} \geq \frac{k}{2}$ of the $\sigma_j(\theta)$ are $=-1$.

For this value of $\theta$ and $\rho$ small enough, $f_t$ with $t = \rho e^{2\pi i \theta}$ has

at least $\frac{k}{2}$ distinct attracting cycles.

To conclude the proof, we prove the claim, by contradiction.

Pick a ray $t(\rho) = \rho e^{2\pi i \theta}$, with $\theta$ chosen so that $t(\rho)$ does not

belong to the finite set of values where $\deg f_t$ drops $(\rho \in [0, +\infty])$

We want to show that the function $t \mapsto z_j(t)$ can be continued analytically

along $t(\rho)$ (or a nbhd of it), still getting a cycle for $f_t$ with constant

multiplier $\lambda_j$.

To do so, we show that the set of $\rho_0 \in [0; +\infty]$ so that $z_j(t(\rho))$ can be

continued for $0 \leq \rho \leq \rho_0$ is both open and closed.

- open, by the implicit function theorem applied to $f_t^{m_j}(z) - z$ at $t = t(\rho_0)$, $z = z_j(t)$.
- closed, because $f_t^{m_j}(z) - z$ defines an algebraic curve on $\hat{\mathbb{C}}_z \times \hat{\mathbb{C}}_t$.

This would imply that $f_\infty(z) = z^d$ has an indifferent cycle.

But the cycles of $f_\infty$ are either superattracting (at $0, \infty$) or repelling

(in $\partial \mathbb{D}$). This gives a contradiction.

$\square$

As a corollary, we obtain:

Corollary. Any rational map $f: \hat{\mathbb{C}} \circlearrowleft$ of degree $d \geq 2$ admits only finitely many non-repelling cycles.

The bound we obtained is $6d-6$. This is not sharp.

For sharp results:

- Douady-Hubbard (1985) For $f: \mathbb{C} \to \mathbb{C}$ polynomial of degree $d \geq 2$, the number of non-repelling cycles is bounded by $d-1$.
(Example: $z^d$ has a cycle at $0$ of multiplicity $d-1$).)

Theorem (Shishikura, 1987). For $f: \hat{\mathbb{C}} \circlearrowleft$ rational map of degree $d \geq 2$, the number of non repelling cycles is bounded by $2d-2$.

Examples: $z \mapsto z^d$ on $\hat{\mathbb{C}}$ has two fixed points of ~~[illegible]~~ multiplier $0$ (at $0, \infty$) of multiplicity $d-1$.

In fact, if $f$ is a polynomial, then $\infty$ is a fixed point of multiplicity $d-1$, and we get again Douady-Hubbard's result for polynomials.

The proof of Shishikura's theorem is based on a new and quite powerful technique to study holomorphic dynamics in one variable, called quasiconformal surgery.

It allows to show also:

- Sullivan non-wandering domain theorem.

- Existence of Herman rings.

(3.5)

Theorem. Let $f: \hat{\mathbb{C}} \circlearrowleft$ be a rational map of degree $\deg f = d \geq 2$.

Then $J(f) = \overline{R(f)}$, where $R(f)$ is the set of repelling cycles.

Proof (Fatou). We want to prove that for any point $z_0 \in J(f)$ and any neighborhood $U$ of $z_0$, there exist $z \in R(f) \cap U$.

Since $J(f)$ has no isolated points, we may assume that $z_0$ does not belong to the finite set $Fix(f) \cup f(\mathcal{C}(f))$ (fixed points $\cup$ critical values).

In other terms, $f^{-1}(z_0) = \{z_1 \dots z_d\}$ is a set of exactly $d$ distinct points ($z_0 \notin f(\mathcal{C}(f))$), all different from $z_0$ ($z_0 \notin Fix(f)$).

By the inverse function theorem, we may locally inverse $f$ at $z_1 \dots z_d$, i.e. find holomorphic maps $\varphi_j : V \to V_j$, $V$ nbhd of $z_0$, $V_j$ of $z_j$, so that $f(\varphi_j(z)) = z$ $\forall z \in V$. ($\varphi_j(z_0) = z_j$). Assume $V, V_j$ all disjoint.

Claim: $\exists n >> 0$, $\exists z \in V$ so that $f^n(z) \in \{z, \varphi_1(z), \varphi_2(z)\}$.

Proof of claim: if not, the maps
$$g_n(z) = \frac{(f^n(z) - \varphi_1(z))(z - \varphi_2(z))}{(f^n(z) - \varphi_2(z))(z - \varphi_1(z))}$$
(i.e. the cross-ratio of $\{f^n(z), z, \varphi_1(z), \varphi_2(z)\}$) on $V$ would avoid the values $\{0, 1, +\infty\}$, which implies $\{g_n|_V\}$ is a normal family.

Since $f^n(z) = \varphi_2(z) + \dfrac{\varphi_2(z) - \varphi_1(z)}{\frac{z - \varphi_1(z)}{z - \varphi_2(z)} g(z) - 1}$, this implies that $\{f^n|_V\}$ is also a normal family, against the hypothesis $z_0 \in J(f)$.

Hence for any $U \ni z_0$ nbhd we may find $n >> 0$ and $z \in U \cap V$ so that either $z = f^n(z)$ or $f^n(z) = \varphi_j(z)$. Hence we get a periodic point $z$ (of period $n$ or $n+1$). This shows that any point in $J(f)$ can be approximated by periodic points.

But since only finitely many periodic points are not repelling, every point in $J(f)$ is approximated by repelling periodic points. □

Consequences:

Corollary: If $U$ is an open set intersecting $J(f)$, then for $\forall n \gg 0$, $f^n(U \cap J(f)) = J(f)$.

Proof: By the theorem, $U$ contains a repelling periodic point $z_0$.
Let $m$ be its period, so that $f^m(z_0) = z_0$.
Pick a smaller nbhd $V \subset U$ of $z_0$ so that $V \subset f^m(V)$ ($z_0$ is repelling).
Then: $V \subset f^m(V) \subset f^{2m}(V) \subset \dots \subset f^{km}(V) \subset \dots$
We already seen that $\bigcup_k f^{mk}(V) \supset J(f^m) = J(f)$. By compactness of $J(f)$, being $(f^{km}(V))_k$ a nested sequence, we get $f^{mk}(V) \supset J(f)$ for some (any) $k \geq k_0$.
Then $f^n(U) \supset f^n(V) \supset J(f)$ for any $n \geq mk_0$. □

Corollary: $\forall U \subset \hat{\mathbb{C}}$ open set, $U \cap J(f) \neq \emptyset$, then no sequence $(f^{n_j})$ can converge locally uniformly on $U$.

Proof: Suppose $f^{n_j} \to g$ locally uniformly. Let $z_0 \in J(f) \cap U$, and $U' \subset U$ neighborhood of $z_0$ so that $|g(z) - g(z_0)| < \varepsilon$ (it suffices: it is bounded).
Then for large $j$, $|f^{n_j}(z) - g(z)| < \varepsilon$ and $|f^{n_j}(z) - g(z_0)| < 2\varepsilon$, against the previous corollary (take $\varepsilon < \frac{\operatorname{diam} J(f)}{2}$). □

We proved previously that Cremer points and boundary of Siegel discs belong to $PC(f)$, where $PC(f)$ is the postcritical set of $f$, $= \bigcup_{n \geq 1} f^n(C(f))$.

Theorem: If $z_0$ is a Cremer point or a boundary point of a Siegel disk, then every neighborhood of $z_0$ contains infinitely many distinct points of $PC(f)$.

In other terms, every such point is an accumulation point of $PC(f)$.

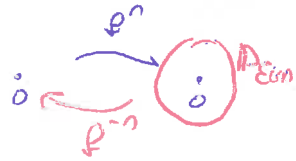
Rem: stronger than the previous result - which could just give $z_0 \in PC(f)$, and not necessarily an accumulation point.

Proof: We first deal with Cremer points.

Up to taking iterates (since $PC(f) = PC(f^m)$), we may assume $z_0 = f(z_0)$ is a Cremer fixed point. We may also assume $z_0 = 0$ up to change of coordinates.

Since $f'(0) \neq 0$, $f$ is locally invertible, and there is a maximal $\varepsilon(n) > 0$ so that the branch of $f^{-n}$ sending $0$ to $0$ is well defined on $\mathbb{D}_{\varepsilon(n)}$.

We can show that, having taken $\varepsilon(n)$ maximal we must have a critical point of $f^n$ on $\partial f^{-n}(\mathbb{D}_{\varepsilon(n)})$




(If not, we can extend locally at any point of $\partial \mathbb{D}_{\varepsilon(n)}$ the map $f^{-n}$, and hence to $\mathbb{D}_\varepsilon$, $\varepsilon > \varepsilon(n)$ by compactness of $\partial \mathbb{D}_{\varepsilon(n)}$).

Equivalently, $\partial \mathbb{D}_{\varepsilon(n)}$ contains a critical value of $f^n$.

We want to show that $\varepsilon(n) \to 0$, which concludes the proof in this case.

Suppose by contradiction that $\exists \varepsilon > 0$, $\varepsilon(n) > \varepsilon$ $\forall n \in \mathbb{N}^*$.

Notice that $f^{-1}(0)$ has at least another point besides $0$.

The grand orbit of $0$ is infinite (or $0$ would be superattracting).

Finally $f^{-n}(\mathbb{D}_{\varepsilon(n)})$ cannot contain $f^{-k}(0) \setminus \{0\}$ $\forall k \leq n$.

It follows that the sequence $(f^{-n}|_{\mathbb{D}_\varepsilon})_n$ is normal.

Some subsequence $f^{-n_j}$ converges to a holomorphic map $g : \mathbb{D}_\varepsilon \xrightarrow{\simeq} U$ with $g(0) = 0$.

Take now $U' = g(\mathbb{D}_{\varepsilon/2})$.

Then for infinitely-many $n$ ($n = n_j$, $j \gg 0$), $f^n(U'_j) \subset \mathbb{D}_\varepsilon$.

This goes against the fact that for all $n$ big enough, $f^n(U')$ contains the whole $J(f)$ (and actually $\bigcup f^n(U') \supseteq \hat{\mathbb{C}} \setminus \mathcal{E}(f)$), since $U' \ni 0 \in J(f)$ ← Cremer point.

We now deal with Siegel disks.

Let $\Delta$ be a fixed Siegel disk ($f(\Delta) = \Delta$).

We have already seen that $\partial\Delta \subset \overline{PC(f)}$. In particular, ~~...~~ any point in $\partial\Delta$ can be approximated by postcritical points; i.e. $PC(f)$ is dense in $\partial\Delta$. In particular, for any point in $\partial\Delta$ and any neighborhood $U$ of it, there are infinitely many points of $PC(f)$ in $U$.

Rem: we can use a similar argument to the Cremer case in the Siegel case to reprove that $PC(f)$ is dense in $\partial\Delta$.

Corollaries: $f : \hat{\mathbb{C}} \circlearrowleft$ of degree $d \geq 2$.

- If every critical orbit of $f$ is either finite or belongs to a contracting basin of attraction, then any cycle of $f$ is either repelling or contracting (i.e., no indifferent cycles).
- If $f$ is post-critically finite ($\# PC(f) < +\infty$), then any cycle is either repelling or superattracting.

~~We will see:~~

- If $f$ is strictly post-critically finite (PCF and all points of $C(f)$ are strictly preperiodic), then ~~...~~ all cycles of $f$ are repelling

(We will see, $\Rightarrow J(f) = \hat{\mathbb{C}}$, from Sullivan's theorem).